How to Calculate Your Personal Safe Withdrawal Rate

Alice laughed. "There's no use trying," she said" One can't believe impossible things."

"I daresay you haven't had much practice," said the Queen. "When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

- From Through the Looking Glass by Lewis Carroll

The recent financial crisis has challenged many comfortably held ideas about investing. Here we examine the assumptions and analytical methods of research about Safe (or "Sustainable") Withdrawal Rate (SWR) and offer a fresh approach to determine the SWR for any investor. The SWR is the maximum, constant fraction of the initial balance of an investment account that can be withdrawn such that the account balance will remain sufficiently large to sustain withdrawals for a time horizon desired by the investor.

There is a wonderful folk theorem in the world of investment advice that says a withdrawal amount of roughly 4% of the initial balance of a retirement account can last the investor almost indefinitely. It appears to us that one source of this idea is Ref{1}, which says:

"Typically, the safe initial withdrawal rate for pretax portfolios is around 4% when all the equities are U.S. large-cap stocks; when smaller-cap stocks are introduced, usually the safe initial withdrawal rate increases modestly to about 4.5%."

How does this idea work? We'll show you in detail, and we won't need to use any historical data. But more on that below.

We provide a new and transparent analysis of the SWR, which has the virtue that it can be updated whenever the investor changes his beliefs about the future of his markets. Our method leads to useful results for investors to directly apply to their decision making whenever they please. The main virtue of our approach is to enable investors to explicitly update their SWR based on new beliefs about their future returns and inflation.

Overview of current SWR Work

The approaches to SWR can be abstracted as follows. An initial balance in an account is assumed, and the investor has a spending goal for a time horizon of N years. The investor withdraws an amount of money equal to the initial balance times F, his initial withdrawal fraction, which amount presumably suffices for the investor's purposes. The problem is to find the maximum F (i.e., SWR) possible for various values of N, or equivalently, the maximum N for a given SWR.

In the above set up, both returns and inflation must be accounted for to determine the evolution of the account balances. For each future year n, there will be a return rate $r^{(n)}$ and inflation rate $i^{(n)}$. (We use the superscript in parenthesis to indicate the time index, as distinguished from an exponent, and employ a subscript to indicate values from a specific set). The previous papers use historical market data, assumptions about portfolio allocations (e.g., 60% stock, 40% bonds) and a variety of simulations and back tests that essentially choose temporal paths of returns and inflation with which to calculate the account yearly balances given a withdrawal rate F. Then the largest F for a sufficiently long time horizon is chosen as the SWR.

The core idea is that the SWR is chosen based upon historical returns and inflation, and certain portfolio asset allocations which support the returns. This framework is inherently backward looking: it takes all the historical data and analysis, generates a set of scenarios (assumptions about which historical data is assumed to be pertinent to the future) and assumes the future is well-represented by the examined price history. Succinctly, the current research has analyzed SWR for back-tested, hypothetical portfolios.

The obvious problem with these SWR approaches is the high likelihood the near future beyond 2010 will present substantially larger return volatility by any measure, due to global deleveraging, recent market panics, massive government spending and high likelihood of inflation. How many investors or advisors would want to use historical portfolio performance data to design long term portfolios? We think not too many. (Actually, the ten year return of S&P 500 through 5 June 2010 is about -25%, and the NASDAQ down about 42%, making history ever more unappealing, and motivating very long planning horizons which have been historically kinder to investors.)

We offer another approach: let the investor incorporate his beliefs about the evolution of his own future returns by his chosen methods, and define the inflation levels he expects. We allow the investor to explicitly capture those beliefs about the future (which beliefs always drive his decisions anyway) to produce a personalized result for SWR. We also rely on an analytical method to derive a formula for SWR in closed form as the basis for our incorporation of the beliefs.

Scenario Based Analysis and Adaptive Decision Making

Our first observation is there is no need to choose a constant withdrawal fraction F at the start of a long investment horizon. Rather, F can be chosen periodically, updating one's financial assumptions as they change. At the start of a specific horizon, our approach computes F, the maximum withdrawal fraction for an assumed horizon of N years, under the assumption of particular return rates and inflation rates over the entire horizon. This set up requires us to calculate the fraction F with which the account depletes after N years, assuming future return and inflation rates. The investor makes his own best guess based on his subjective and specific *beliefs* about all factors that would produce returns and inflation for his own situation.

Constant Rates, Scenarios and Paths

The investor looks ahead for an arbitrary number n of periods, where n≥ 0. Each period n has a return rate $r^{(n)}$ and inflation rate $i^{(n)}$. While clearly this is a realistic assumption, it leads to cumbersome analysis. Fortunately, we can show that any series of returns and inflation can be replaced by an equivalent constant return or inflation rate for the whole horizon. Thus, we initially assume constant return rates $r^{(n)} = r$ and inflation rates $i^{(n)} = i$ are known for all periods n. (We relax this assumption later.) The constant values r and i produce the same end results as any series of variable rates (see the proof below in Appendix 1). So we will operate under an analytical assumption of constant periodic returns and inflation chosen equivalent to the respective, realized time series. The only caveat is this: if the return over any period is less than -100%, the investor's account drops below $0 and the process stops. In our model we allow for any periodic returns equal to or larger than -100%.

To further account for reality, we allow the investor to evaluate and commit to his own views of his future. Thus, our model structures the assumptions each investor inherently makes and captures them in a useful way. The investor has no need to embrace any particular historical interpretations or predictions based on them, although we recognize that all predictions implicitly rely upon possibilities revealed by history. We simply define a range of possible values of return rates and inflation rates, and then specify a probability distribution for those values. Each distribution, by its probability weighting, defines a scenario. A pair of return rate and inflation rate scenarios, as defined by their distributions, combine to form a "path".

These paths derive from different beliefs about the rates that will occur. Each path describes a belief about a future state of the world that produces a specific pair of r and i distributions. The issue here is treating the randomness of r and i, not their constancy. A path encapsulates an entire future "world" as represented by the joint probability distribution of r and i.

To illustrate this idea, assume five possible values for the future returns:

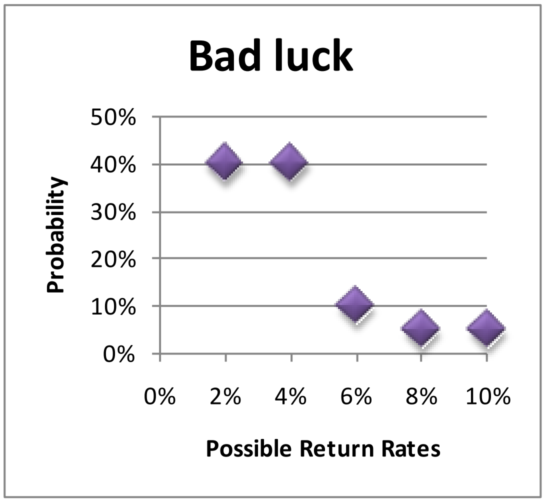
{2%, 4%, 6%, 8%, 10% }.

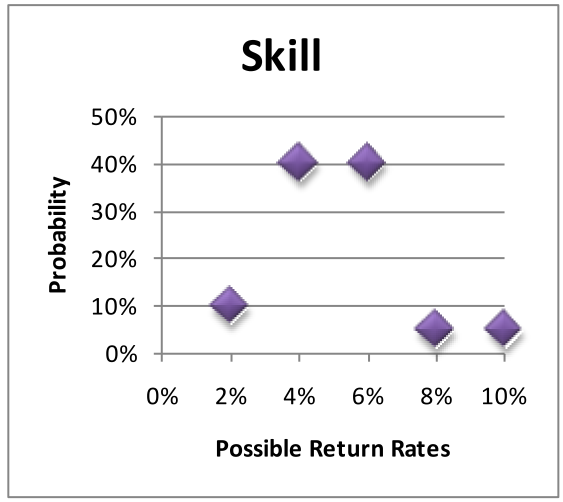
Also assume the same possible values for inflation, though in the model we can choose any number and any values of return rates and inflation rates.

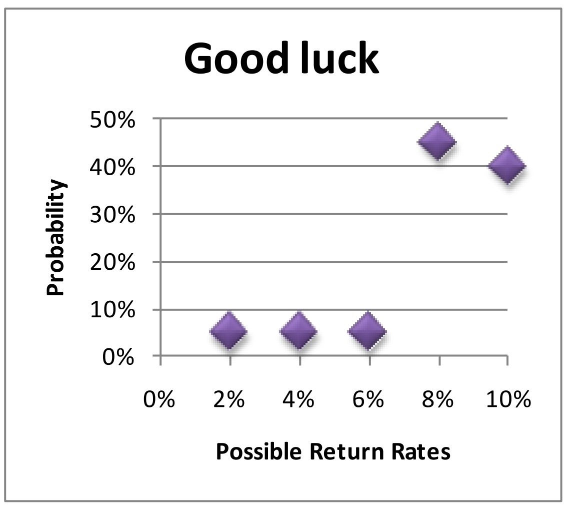
We capture our beliefs about the paths under different conditions by assigning subjective probabilities that each scenario value actually occurs over the time horizon. Each particular path, comprised by definition of two scenarios, has a total in this example of 5x5 possible pairs of fixed return rates and inflation rates.

Further, to consider a wider view of our future, we choose three return rate scenarios ("Bad Luck", "Skill", "Good Luck") and three inflation scenarios ("Minimal", "Observable", "Runaway"). Then there are nine possible paths, each defined by the specified pair of scenario distributions.

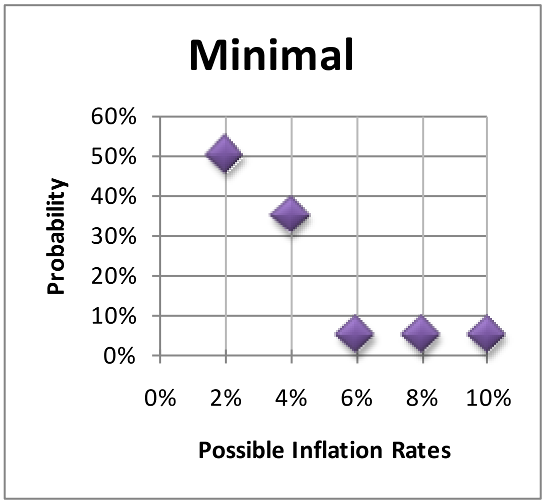
The three return rate scenarios are charted here:

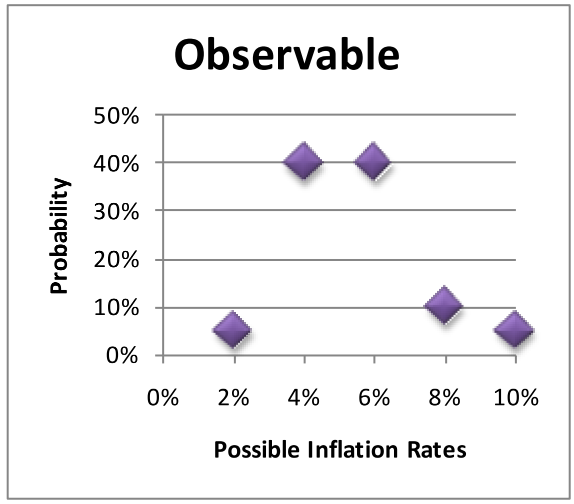


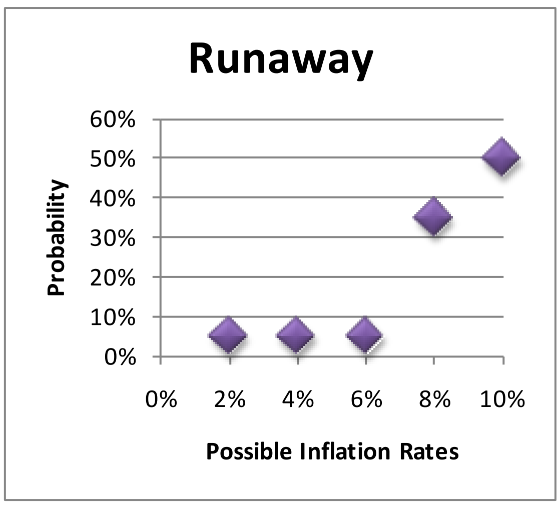




The three inflation rate scenarios are plotted here:







Formal Definitions

We require the investor to assume a discrete, joint distribution of return and inflation rates and to limit the possibilities for outcomes of r and i values into some finite number of values for each:

$r = r_j$, $j = 1, 2, \ldots L1$

$i = i_k$, $k = 1, 2, \ldots L2$

where r_j means the $j - th$ realized future value of a return rate scenario over the time horizon of n years. Similarly, i_k means the $k - th$ realized future value of an inflation rate over the n period horizon. He must assign his subjective value of the probability of occurrence of joint event $[r = r_j, i = i_k]$ by specifying the probability

$P[r = r_j, i = i_k]$ $for\ r = r_j$, $j = 1, 2, \ldots L1$; $i = i_k$, $k = 1, 2, \ldots L2$

We assume the subjective joint probability distribution for r and i over the time horizon is well-modeled by the product of independent distributions. This assumption has some empirical support in Ref{5}, which says:

"Our general conclusion is that it is extremely hard to define hard and fast rules with respect to the link between inflation and equity returns."

Thus, we may write

$P[r = r_j, i = i_k] = P[r = r_j] * P[\ i = i_k]$

For example, with $L1 = L2 = 5$, we could assume r will take any one of these 5 values as a constant return: {2%, 4%, 6%, 8%, 10%}.This example captures the investor's belief, for whatever reason, that over the future horizon of interest, returns will be at least 2%, but will not exceed 10%.

Further, we could take the same set for possible values of inflation i with the assumption that inflation will never be below 2%, nor permitted to exceed 10%.

Our innovation

In this section we will use

$F[r_j, i_k\ | n = N]$

to mean the value of F for a given number of periods N as a direct function of values $r = r_j, i = i_k$. Sometimes we will call F the Point Withdrawal Fraction.

We derive F analytically below.

Our innovation is to compute the expectation $\bar{F}$ of F over the chosen scenarios, so that

$$\bar{F}(N) = E[F[r, i \,|\text{n} = \text{N}]]$$

$$\bar{F}(N) = \sum_{j=1}^{L1} \quad \sum_{k=1}^{L2} F\left[r_j, i_k \,\middle|\text{n} = \text{N}\right] * \text{P}[r = r_j, i = i_k]$$

where the expectation is taken over the set of investor-defined scenarios with investor-defined probabilities described above. The averaged value $\bar{F}(N)$ of F explicitly accounts for the investor's subjective probabilities of his own beliefs about future returns and inflation over arbitrary horizon N and captures the joint effects of the inflation and return scenarios in one number.

Using the practical assumption that returns are independent of inflation, as noted above, we can calculate the path withdrawal fraction $\bar{F}(N)$

$$\bar{F}(N) = \sum_{j=1}^{L1} \quad \sum_{k=1}^{L2} F\left[r_j, i_k \,\middle|\text{n} = \text{N}\right] * \text{P}\left[r = r_j\right] * P[\, i = i_k]$$

Thus, by capturing the investor's own beliefs, he can compute the maximum draw down rate for any assumed time horizon N. When news or contemplation alter his beliefs about his future, he can very easily define new scenarios. Note that there is one path withdrawal fraction $\bar{F}(N)$ for each path analyzed.

Derivation of Equations

More Definitions

B_n =Account balance at end of period n; $n \geq 0$

B_0 = Initial balance at start of period 1

r= investment rate of return per period n; $n \geq 0$; $r \geq 0$
i= inflation rate per period $n, n \geq 0$; $i \geq 0$

$I = 1 + i$ = gross inflation
$R = 1 + r$ = gross return
$q = \frac{I}{R}$

F = fraction of initial balance B_0 withdrawn

W_0 = FB_0 = initial amount withdrawn

W_n= amount of account withdrawn at end of period $n, n \geq 0$

This function will be useful below:

$\mathrm{G_n(q) = n,\ \ if\ q = 1}$

$$\mathrm{G_n\,(q) = \frac{1 - q^n}{1 - q}\,,\text{if } q \neq 1}$$

It can be shown that

$\mathrm{G_n}\,(\mathrm{q}) = \sum_{k=0}^{n-1} q^k$

WITHDRAWAL PROTOCOL

The sequence of withdrawals evolves as illustrated in the table:

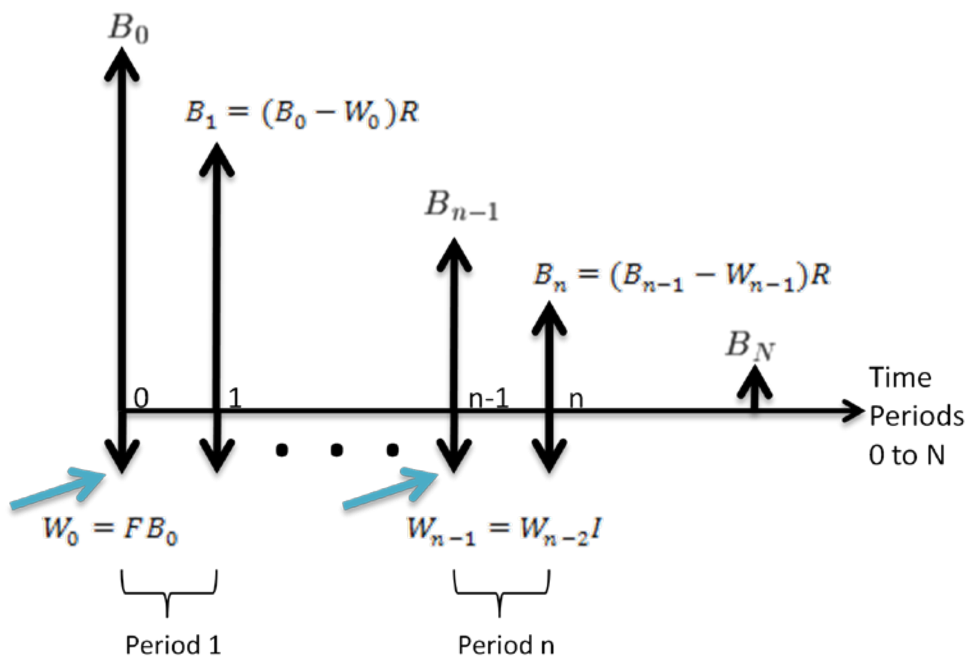
End of Time Period	**Account Balance at end of period, after Withdrawal**	**Amount withdrawn at end of period**
0	B_0	
1	$B_1=(B_0-W_0)R$	$W_0=FB_0$
2	$B_2=(B_1-W_1)R$	$W_1=W_0I$
•		
•		
n	$B_n=(B_{n-1}-W_{n-1})R$	$W_{n-1}=W_{n-2}I$

The account begins period 1 with amount B_0. At the start of period 1, W_0 is withdrawn. Balance B_1 is amount B_0 reduced by W_0 and then grown at rate r. This process evolves through n periods, for n = 1,2,3,....

When inflation i=0, the withdrawals would be constant by design. But when inflation is positive, withdrawals must grow at the inflation rate to keep the amounts withdrawn equal in constant dollars referenced to time n=0.

The $\{B_n\}$ in the table are depicted as the vertical arrows in the diagram below.

Evolution of Account Balances



We can solve the recursions in the table. First, we have

$$W_n = W_0 I^n, n \geq 0 \qquad [1]$$

We use equation [1] in the main recursion in bottom row of the table as follows:

$$B_n = (B_{n-1} - W_0 I^{n-1})R, \quad n \geq 1 \qquad [2]$$
$$= RB_{n-1} - FB_0 R I^{n-1}, \quad n \geq 1 \qquad [3]$$

Solution to General Recursion

To see the solution to [3], consider the generic variables in a recursion

$$Y_n = Y_0, n = 0$$

$$Y_n = H Y_{n-1} + V_n, \quad n \geq 1 \qquad [4]$$

The solution to [4] can be verified to be:

$$Y_n = Y_0 H^n + \sum_{k=1}^{n} V_k H^{n-k}, \quad n \geq 1 \qquad [5]$$

After applying this generic solution [5] to the formula [3] for B_n we find

$$B_n = R^n B_0 - \sum_{k=1}^{n} F B_0 R I^{k-1} R^{n-k} \qquad n \geq 1$$

$$= B_0 R^n (1 - F \sum_{m=0}^{n-1} q^m) \qquad n \geq 1 \qquad [6]$$

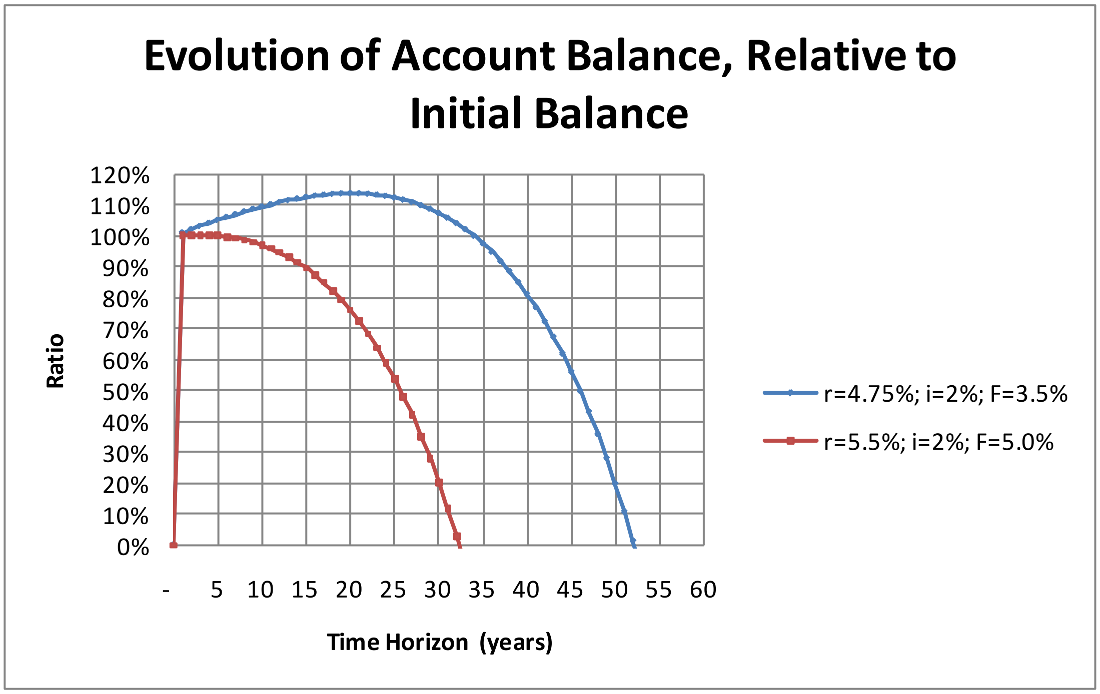
The summation above can be simplified and the result shown to be

$$B_n = B_0 R^n (1 - F G_n (q)); \qquad n \geq 1 \qquad [7]$$

This equation shows how the account balances evolve over time for particular values of r and i for all periods. The relationship among F, r and i can be subtle, as we will explain. The function $G_n (q)$ was defined in the Definitions section above.

This chart plots $\frac{B_n}{B_0}$ for $n \geq 0$.

Example of Normalized Balances



This graph uses the parameters specified in the top legend. Notice how inflation rate is the same for both curves, but the curve with the higher return rate gets much lower number of periods before account reaches zero. This is because the value of F is larger for the high return case, and so the account is drawn down faster than it grows, compared to the low return situation with smaller withdrawal rate. This simply shows some of the inter-relations between F, and rates of return and inflation.

Safe Withdrawal Rate: An "Optimum" Value of F

There must be an optimum value of F, which we have defined to be the Safe Withdrawal Rate (SWR). The reasoning is straightforward. Referring to equation [7] above, if F is small enough, then the balances $\{B_n\}$ can remain positive for as many periods as we wish, but then our amounts withdrawn likely will be insufficient to support lifestyle. As F grows towards 1, lifestyle may be better supported, but the balances will drop to 0 very fast. So we seek some value of F, called the SWR, that forms a useful compromise between the extremes of F too small (near 0) and too large (near 1).

First, observe that to ensure $B_n \geq 0,\ n \geq 0$, it suffices that

$$1 - FG_n(q) \ > 0$$

from which we get

$$F \ < \frac{1}{G_n\,(q)} \qquad [8]$$

where we use the defined terms

$$\frac{1}{G_n\,(q)} = \frac{1}{n}\,, \qquad for\ q = 1 \qquad [9a]$$

and

$$\frac{1}{G_n\,(q)} = \frac{1-q}{1-\ q^n}\,, \qquad for\ q\ \neq 1 \qquad [9b]$$

Case 1: Inflation matches the return rate ($\boldsymbol{i = r}$)

In this case, q=1, because inflation rate matches the return rate. Thus we can arbitrarily choose n=N periods for the duration of the account, and also choose

F< 1/N

With equality above, F will require equal withdrawals of B_0/N and will give a zero balance at end of period N.

Case 2: General condition with inflation different from return rate ($\boldsymbol{i \neq r}$)

If we want some F for which B_n> 0 for an arbitrary number n of periods, F is bounded from above by [9b]. In this case, the account will not completely empty for n periods, but F may well be too small for lifestyle. So let's see if we can make F large enough to have bigger balances to enable larger withdrawals for lifestyle purposes, accepting a possibly sooner account depletion.

We now seek the maximum number of periods N for which balances remain positive, and after which, using withdrawals as specified in the protocol table, the balance would become negative. This value for n=N would constrain the point withdrawal rate F. Returning to formula [7], we seek N such that

$$B_N = 0, \qquad [10]$$

It follows that we must solve for N in [7] to satisfy [10]:

$$1 - FG_N(q) = 0 \qquad [11a]$$

It is easy to find

$$F[r_j, i_k \,|\mathrm{N}] = 1/G_N(q) \qquad [11b]$$

where we use as before

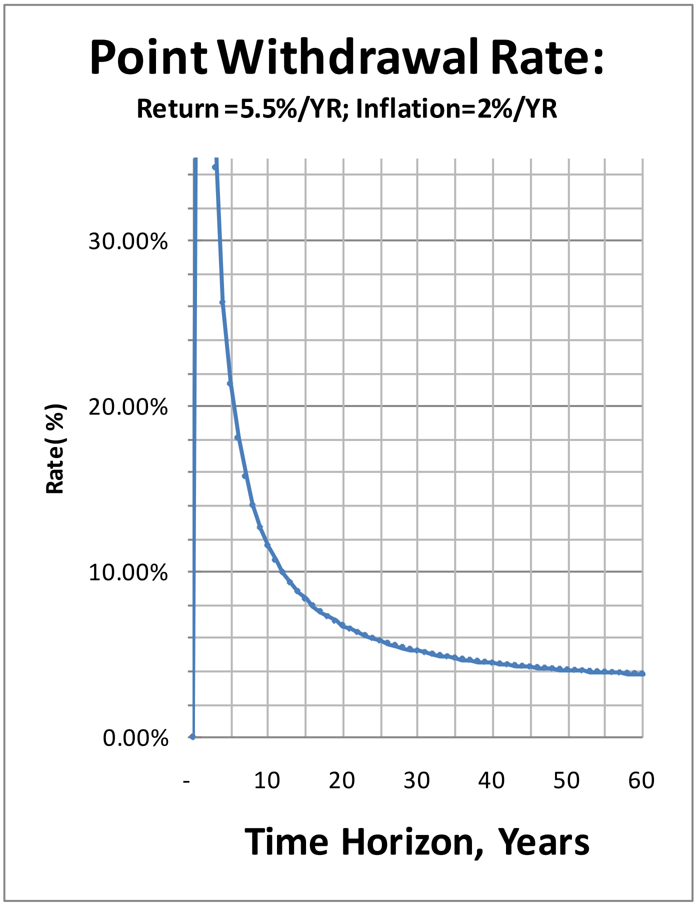
$$q = \frac{1+i}{1+r} = \frac{(1+i_k)}{(1+r_j)} \qquad [11c]$$

and equation [9a, 9b]. For given values of r_j and i_k we have

$$SWR = F[r_j\,, i_k\,|N]$$

Formula [11b] is plotted here:

Example: Point Withdrawal Fraction



APPENDIX 1:THERE IS ALWAYS A CONSTANT RETURN RATE EQUIVALENT TO ANY SERIES OF VARIABLE RETURNS

We write the equation for growth of account balances thus:

$$B_n = B_{n-1}R_{n-1} + V_n \ \ for \ \ n \geq 1 \qquad [12]$$

We have used a periodic gross return defined as

$$R_n = \left(1 + r^{(n)}\right) for \ \ n \geq 0 \qquad [13]$$

We use $r^{(n)}$ as the periodic return rate. The solution to (12) can be found by induction:

$$B_n = B_0\widetilde{R}\,(n) + \sum_{m=1}^{n} V_m\,\rho(m,n) \qquad [14]$$

where we have defined the functions

$$\tilde{R}(n) = \textstyle\prod_{k=0}^{n-1} R_k \text{ for n } \geq 1 \qquad [15]$$

$$\rho(m,n) = \prod_{l=m}^{n-1} R_l \qquad for \ \ m \leq n-1 \qquad [16a]$$

$$\rho(m,n) = 1 \qquad for\, m > n-1 \qquad [16b]$$

By solving this equality

$$R^n = \tilde{R}(n)$$

we find the equivalent constant rate of gross return and the constant return rate:

$$R = [\prod_{k=0}^{n-1}(1 + r^{(k)})]^{\frac{1}{n}} \qquad [17]$$

and

$$r = R - 1 \qquad [18]$$

APPENDIX 2: APPROXIMATE DEPENDENCE ON EXCESS RETURN OVER INFLATION

It's intuitive to believe the SWR should depend only on the excess return over inflation. We define this excess return as Δ:

$$\Delta = \mathrm{r} - \mathrm{i}$$

Consider the identity

$$\frac{1}{1+r} = 1 - r/(1+r)$$

We will expand the right side above, apply recursion and collect higher order terms and use this result to calculate q and its approximation:

$$\frac{1}{1+r} = 1 - r + r^2 - U(r)$$

Where we defined

$$U(r) = r^3/(1+r)$$

We now can find

$$q = (1+i)(1-r) + r^2(1+i) - (1+i)U(r)$$

$$= 1 - \Delta + r\Delta + ir^2 - (1+i)U(r)$$

For values of $r \ll 1$ and $i \ll 1$, this expression reduces to the approximation:

$$q \sim 1 - \Delta$$

which is what we sought. From a practical point of view, we can take "$\ll$" as "< 0.1".

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